

EFFECTIVE JOINT DISTRIBUTION OF EIGENVALUES OF HECKE OPERATORS

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ABSTRACT. In 1997, Serre proved that the eigenvalues of normalised p -th Hecke operator T'_p acting on the space of cusp forms of weight k and level N are equidistributed in $[-2, 2]$ with respect to a measure that converge to the Sato-Tate measure, whenever $N+k \rightarrow \infty$. In 2009, Murty and Sinha proved the effective version of Serre's theorem. In 2011, using Kuznetsov trace formula, Lau and Wang derived the effective joint distribution of eigenvalues of normalized Hecke operators acting on the space of primitive cusp forms of weight k and level 1. In this paper, we extend the result of Lau and Wang to space of cusp forms of higher level. Here we use Eichler-Selberg trace formula instead of Kuznetsov trace formula to deduce our result.

1. INTRODUCTION

Let $S(N, k)$ be the space of all holomorphic cusp forms of weight k with respect to $\Gamma_0(N)$. For any positive integer n , let $T_n(N, k)$ be the n -th Hecke operator acting on $S(N, k)$. Let $s(N, k)$ denote the dimension of the space $S(N, k)$. For a positive integer $n \geq 1$, let

$$\{\lambda_i(n)\}, 1 \leq i \leq s(N, k)$$

denote the eigenvalues of T_n , counted with multiplicity. For any positive integer n , let

$$T'_n := \frac{T_n}{n^{\frac{k-1}{2}}}$$

be the normalized Hecke operator acting on $S(N, k)$ with eigenvalues

$$\left\{ a_i(n) = \frac{\lambda_i(n)}{n^{\frac{k-1}{2}}}, 1 \leq i \leq s(N, k) \right\},$$

counted with multiplicity. By the celebrated theorem of Deligne [4], which proves the famous Ramanujan conjecture, we know that for any prime p , such that p coprime to N , the eigenvalue of T'_p lies in the interval $[-2, 2]$. The recently proved Sato-Tate conjecture by Barnet-Lamb, Geraghty, Harris and Taylor [1], [2] and [5] says that if $a_i(p)$, $1 \leq i \leq r$ is a p -th normalized Hecke eigenvalue, then the family $\{a_i(p)\}$ is equidistributed in $[-2, 2]$ as $p \rightarrow \infty$ with respect to the Sato-Tate measure

$$d\mu_\infty = \frac{1}{2\pi} \sqrt{4 - x^2} dx.$$

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More precisely, the Sato-Tate conjecture states that for any continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, positive integer V and any interval $[\alpha, \beta] \subset [-2, 2]$

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_{p \leq V} \phi \left(\frac{a_i(p)}{p^{\frac{k-1}{2}}} \right) = \int_{\alpha}^{\beta} \phi d\mu_{\infty}.$$

In 1997, Serre [11] studied the “vertical” Sato-Tate conjecture by fixing a prime p and varying N and k . In particular, he proved the following theorem:

Theorem 1. *Let p be a prime number. Let $\{(N, k)\}$ be a sequence of pairs of positive integers such that k is even and p is coprime to N and $N + k \rightarrow \infty$. Then the family of eigenvalues of the normalized p -th Hecke operator*

$$T'_p(N, k) = \frac{T_p(N, k)}{p^{\frac{k-1}{2}}}$$

is equidistributed in the interval $\Omega = [-2, 2]$ with respect to the measure

$$\mu_p := \frac{p+1}{\pi} \frac{\sqrt{1 - \frac{x^2}{4}}}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - x^2} dx$$

Remark 2. *Also in 1997, Conrey, Duke and Farmer [3] studied a special case of above result by fixing $N = 1$.*

In 2009, Murty and Sinha [8] investigated the effective/quantitative version of Serre’s results, in which they give explicit estimate on the rate of convergence. They proved the following theorem

Theorem 3. *Let p be a fixed prime. Let $\{(N, k)\}$ be a sequence of pairs of positive integers such that k is even, p is coprime to N . For an interval $[\alpha, \beta] \subset [-2, 2]$,*

$$\frac{1}{s(N, k)} \# \{1 \leq i \leq s(N, k) : a_i(p) \in [\alpha, \beta]\} = \int_{\alpha}^{\beta} \mu_p + O \left(\frac{\log p}{\log kN} \right).$$

As a continuation of their paper, in 2010, Murty and Sinha [8] proved a quantitative equidistribution theorem for the eigenvalues of Hecke operators acting on the space $S^{new}(N, k)$. Recently Lau and Wang [7] computed the rate of convergence in Sarnak’s [10] result using the Kuznetsov trace formula. Indeed they proved the joint distribution of eigenvalues of the Hecke operators quantitatively for primitive Maass forms of level 1 and stated that the same hold true for primitive holomorphic cusp forms. More precisely, they proved the following theorem:

Theorem 4. *Let p_1, p_2, \dots , and p_r be distinct primes. Let k be a positive even integer such that $r \log(p_1 p_2, \dots, p_r) \leq \delta \log k$, for some small absolutely constant δ . Let $a'_i(p_i)$ be the eigenvalues of normalized Hecke operators T'_{p_i}*

acting on $S^{new}(1, k)$. For any $I = [\alpha_n, \beta_n]^r \subset [-2, 2]^r$

$$\frac{\#\left\{1 \leq n \leq s(1, k) : \left(a_1^{(n)}(p_1), \dots, a_r^{(n)}(p_r)\right) \in I\right\}}{s^{new}(1, k)} \\ = \int_I \prod_{n=1}^r d\mu_{p_n} + O\left(\frac{r \log(p_1 p_2 \dots p_r)}{\log k}\right),$$

$$\text{where } d\mu_{p_n} = \frac{p_n + 1}{\pi} \frac{\sqrt{1 - \frac{x^2}{4}}}{\left(p_n^{\frac{1}{2}} + p_n^{-\frac{1}{2}}\right)^2 - x^2} dx.$$

They have remarked that their methods do work for primitive forms in higher level. In this paper we extend their result to the cusp forms of any level N using ‘‘Eichler-Selberg trace formula’’. Precisely, we prove the following theorem:

Theorem 5. *Let p_1, p_2, \dots , and p_r be distinct primes. Let k be positive even integer such that $r \log(p_1 p_2, \dots, p_r) \leq \delta \log k$, for some small absolutely constant δ . For $1 \leq i \leq r$, let $a_i^{(n)}(p_i), 1 \leq n \leq s(N, k)$ be the eigenvalues of normalized Hecke operators T'_{p_i} acting on $S(N, k)$. For any $I = [\alpha_n, \beta_n]^r \subset [-2, 2]^r$*

$$\frac{\#\left\{1 \leq n \leq s(N, k) : \left(a_1^{(n)}(p_1), \dots, a_r^{(n)}(p_r)\right) \in I\right\}}{s(N, k)} \\ = \int_I \prod_{n=1}^r d\mu_{p_n} + O\left(\frac{r \log(p_1 p_2 \dots p_r)}{\log(kN)}\right),$$

$$\text{where } d\mu_{p_n} = \frac{p_n + 1}{\pi} \frac{\sqrt{1 - \frac{x^2}{4}}}{\left(p_n^{\frac{1}{2}} + p_n^{-\frac{1}{2}}\right)^2 - x^2} dx \text{ and the implied constant is effectively computable.}$$

2. EQUIDISTRIBUTION AND ITS EXTENSION

A sequence of real numbers $\{x_n\}_{n=1}^\infty$ is said to be uniformly distributed or (equidistributed) (mod 1) if for any interval $[\alpha, \beta] \subset [0, 1]$, we have

$$\lim_{V \rightarrow \infty} \frac{1}{V} \#\{n \leq V : x_n \in [\alpha, \beta]\} = \beta - \alpha.$$

In 1916, Weyl [12] proved the following criterion for uniform distribution (mod 1). A sequence $\{x_n\}_{n=1}^\infty$ is uniformly distributed if and only if for any integer $m \neq 0$

$$\sum_{n \leq V} e(mx_n) = o(V) \text{ as } V \rightarrow \infty.$$

Since the set of trigonometric polynomials is dense in $C^1[0, 1]$, the above criterion of Weyl is equivalent to the assertion that, for any continuous

function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_{n \leq V} f(x_n) = \int_0^1 f(t) dt.$$

A sequence of tuples $\{(x_1^{(n)}, x_2^{(n)}, \dots, x_r^{(n)})\}$ in \mathbb{R}^r is said to be uniformly distributed or (equidistributed) (mod 1) if for every $I = \prod_{n=1}^r [\alpha_n, \beta_n] \subset [0, 1]^r$, we have

$$\lim_{V \rightarrow \infty} \frac{\#\{n \leq V : (x_1^{(n)}, x_2^{(n)}, \dots, x_r^{(n)}) \in I\}}{V} = \mu(I),$$

where μ is the usual Lebesgue measure on \mathbb{R}^r . In the same paper Weyl [12] extended his criterion of equidistribution to the higher dimension as follows: A sequence of tuples $\{(x_1^{(n)}, x_2^{(n)}, \dots, x_r^{(n)})\}$ in \mathbb{R}^r is uniformly distributed if and only if for any integers $m_1, m_2, \dots, m_r \neq 0$

$$\sum_{n=1}^V e\left(\sum_{i=1}^r m_i x_n^{(i)}\right) = o(V) \text{ as } V \rightarrow \infty.$$

Note that as in the one variable case, the set of trigonometric polynomials is also dense in $C([0, 1]^r)$, the above criterion is equivalent to the following statement:

A sequence of tuples $\{(x_1^{(n)}, x_2^{(n)}, \dots, x_r^{(n)})\}$ in \mathbb{R}^r is uniformly distributed if and only if for any continuous function $f : \mathbb{R}^r \rightarrow \mathbb{R}$,

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_{n \leq V} f(x_1^{(n)}, x_2^{(n)}, \dots, x_r^{(n)}) = \int_I f(x_1, x_2, \dots, x_r) dx_1 dx_2 \cdots dx_r.$$

Now we define the set equidistribution as follows:

A sequence of finite multi sets A_n with $\#A_n \rightarrow \infty$ is said to be set equidistributed (mod 1) with respect to a probability measure μ if for every continuous function $f \in C^1[0, 1]$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\#A_n} \sum_{t \in A_n} f(t) = \int_0^1 f(x) d\mu.$$

The criterion of Schoenberg and Wiener says that the sequence $\{A_n\}$ is set equidistributed with respect to some positive continuous measure if and only if the Weyl limit

$$c_m := \lim_{n \rightarrow \infty} \frac{1}{\#A_n} \sum_{t \in A_n} e(mt)$$

exists and

$$\sum_{m=1}^V |c_m|^2 = o(V).$$

For our purpose, let us define the set equidistribution in higher dimension. A tuples of finite multi set say $\Omega = (A_1^{(n)}, A_2^{(n)}, \dots, A_r^{(n)})$ with $\#A_i^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$ for all $i = 1, 2, \dots, r$ is said to be set a set equidistributed (mod

1) with respect to a probability measure μ if for every continuous function $f \in C^1([0, 1]^r)$, and $I = [0, 1]^r$ we have

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_{n \leq V} f(x_1, x_2, \dots, x_r) = \int_I f(x_1, x_2, \dots, x_r) dx_1 dx_2 \cdots dx_r.$$

With this generalization, we define the Weyl limit as

$$C_m := \lim_{n \rightarrow \infty} \frac{1}{\prod_{i=1}^r \#A_n} \sum_{(t_1, t_2, \dots, t_r) \in \Omega} e(m_1 t_1 + m_2 t_2 + \cdots + m_r t_r).$$

3. EICHLER SELBERG TRACE FORMULA AND ITS ESTIMATIONS

In this section, we use Eichler-Selberg trace formula, as one of our important tool to prove the main theorem, which is a formula for the trace of T_n acting on $S(N, k)$ in terms of class number of binary quadratic forms and few others arithmetic functions. We follow the presentation of [8]. For a non-negative integer $\Delta \equiv 0, 1 \pmod{4}$, let $B(\Delta)$ be the set of all positive definite binary quadratic forms with discriminant Δ . That is

$$B(\Delta) = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, a > 0, b^2 - 4ac = \Delta\}.$$

We denote the set of primitive forms by

$$b(\Delta) = \{f(x, y) \in B(\Delta) : \gcd(a, b, c) = 1\}.$$

Now we define an action of the full modular group $SL_2(\mathbb{Z})$ on $B(\Delta)$ as follows:

$$f(x, y) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := f(\alpha x + \beta y, \gamma x + \delta y).$$

Note that the above action takes primitive forms to primitive forms. We know that the above action has finitely many orbits. We define $h(\Delta)$ to be the number of orbits of $b(\Delta)$. Let h_w be defined as follows:

$$\begin{aligned} h_w(-3) &= \frac{1}{3}, \\ h_w(-4) &= \frac{1}{2}, \\ h_w(\Delta) &= h(\Delta) \text{ for } \Delta < -4. \end{aligned}$$

We define some arithmetical functions which are going to useful to state the Eichler-Selberg trace formula.

Let

$$A_1(n) = \begin{cases} n^{\frac{k}{2}-1} \left(\frac{k-1}{12}\right) \psi(N) & \text{if } n \text{ is a square} \\ 0 & \text{otherwise,} \end{cases}$$

where $\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$. Let

$$A_2(n) = -\frac{1}{2} \sum_{\substack{t \in \mathbb{Z} \\ t^2 < 4n}} \frac{\varrho^{k-1} - \bar{\varrho}^{k-1}}{\varrho - \bar{\varrho}} \sum_g h_w \left(\frac{t^2 - 4n}{g^2} \right) \mu(t, g, n),$$

where ϱ and $\bar{\varrho}$ are the zeros of the polynomial $x^2 - tx + n$, the inner sum runs over positive divisors g of $\frac{(t^2-4n)}{g^2} \in \mathbb{Z}$ is congruent to 0 or 1 (mod 4) and $\mu(t, g, n)$ is given by

$$\mu(t, g, n) = \frac{\psi(N)}{\psi(\frac{N}{N_g})} M(t, n, NN_g),$$

with $N_g = \gcd(N, g)$ and $M(t, n, k)$ denote the number of solutions of the congruence $x^2 - tx + n \equiv 0 \pmod{K}$. Now, we let

$$(1) \quad A_3(n) = - \sum_{\substack{d|n \\ 0 < d \leq \sqrt{n}}} d^{k-1} \sum_{c|N} \phi \left(\gcd \left(c, \frac{N}{c} \right) \right),$$

where ϕ denotes the Euler's function and in the first summation, if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $\frac{1}{2}$. In the inner sum, we also need the condition that $\gcd(c, \frac{N}{c})$ divides $\gcd(N, \frac{n}{d} - d)$. Finally, let

$$A_4(n) = \begin{cases} \sum_{\substack{t|n \\ t > 0}} t & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6. *For any positive integer n , let $\text{Tr}(T_n)$ be the trace of T_n acting on $S(N, k)$. Then, we have,*

$$\text{Tr}(T_n) = A_1(n) + A_2(n) + A_3(n) + A_4(n).$$

We use the above Eichler-Selberg trace formula to prove the following Proposition:

Proposition 7. *For any positive integers m_1, m_2, \dots, m_r we have,*

$$|\text{Tr}(T_{p_1^{m_1} \dots p_r^{m_r}})| \ll p_1^{\frac{3m_1}{2}} \dots p_r^{\frac{3m_r}{2}} (m_1 \dots m_r) d(N) \sqrt{N} \log(4(p_1^{m_1} \dots p_r^{m_r})).$$

Proof. Consider

$$\begin{aligned} \text{Tr}(T_{p_1^{m_1} \dots p_r^{m_r}})' &= B_1(m_1 m_2 \dots m_r) + B_2(m_1 m_2 \dots m_r) + B_3(m_1 m_2 \dots m_r) \\ &\quad + B_4(m_1 m_2 \dots m_r), \end{aligned}$$

where $B_i(m_1 m_2 \dots m_r) = \frac{A_i(p_1^{m_1} \dots p_r^{m_r})}{(p_1^{m_1} \dots p_r^{m_r})^{\frac{k-1}{2}}}$, for all $i = 1, 2, 3, 4$. We use the estimates of [8] and get the estimates for each $B_i(m_1 m_2 \dots m_r)$, for all $i = 1, 2, 3, 4$. First, we consider $i = 1$

$$\begin{aligned} B_1(p_1^{m_1} \dots p_r^{m_r}) &= \frac{A_1(p_1^{m_1} \dots p_r^{m_r})}{(p_1^{m_1} \dots p_r^{m_r})^{\frac{k-1}{2}}} \\ &= \begin{cases} \frac{(p_1^{-\frac{m_1}{2}} \dots p_r^{-\frac{m_r}{2}})^{\frac{k-1}{12}}}{(p_1^{m_1} \dots p_r^{m_r})^{\frac{k-1}{2}}} & \text{if } m_1, m_2, \dots, m_r \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} (p_1^{-\frac{m_1}{2}} \dots p_r^{-\frac{m_r}{2}})^{\frac{k-1}{12}} & \text{if } m_1, m_2, \dots, m_r \text{ are even,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now, we shall consider $i = 2$ as follows.

$$\begin{aligned} B_2(p_1^{m_1} \cdots p_r^{m_r}) &= \frac{A_2(p_1^{m_1} \cdots p_r^{m_r})}{(p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}} \\ &= \frac{8e}{\log 2} (2) p_1^{\frac{3m_1}{2}} \cdots p_r^{\frac{3m_r}{2}} 2^{\nu(N)} \log 4(p_1^{m_1} \cdots p_r^{m_r}). \end{aligned}$$

For $i = 3$, we get,

$$\begin{aligned} B_3(p_1^{m_1} \cdots p_r^{m_r}) &= \frac{A_3(p_1^{m_1} \cdots p_r^{m_r}) d(N) \sqrt{N}}{(p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}} \\ &\leq \frac{d(p_1^{m_1} \cdots p_r^{m_r}) (p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}} d(N) \sqrt{N}}{(p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}} \\ &\leq (m_1 + 1) \cdots (m_r + 1) d(N) \sqrt{N}. \end{aligned}$$

Finally, for $i = 4$, we get,

$$\begin{aligned} B_4(p_1^{m_1} \cdots p_r^{m_r}) &= \frac{A_4(p_1^{m_1} \cdots p_r^{m_r})}{(p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}} = \begin{cases} \frac{d(p_1^{m_1} \cdots p_r^{m_r}) (p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}}{(p_1^{m_1} \cdots p_r^{m_r})^{\frac{k-1}{2}}} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} p_1^{\frac{m_1}{2}} \cdots p_r^{\frac{m_r}{2}} (m_1 + 1) \cdots (m_r + 1) & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Combining all the estimates above, we get

$$\begin{aligned} &|Tr (T_{p_1^{m_1} \cdots p_r^{m_r}})| \\ &\ll (p_1^{-\frac{m_1}{2}} \cdots p_r^{-\frac{m_r}{2}})^{\frac{k-1}{2}} + p_1^{\frac{3m_1}{2}} \cdots p_r^{\frac{3m_r}{2}} \log 4(p_1^{m_1} \cdots p_r^{m_r}) \\ &+ (m_1 + 1) \cdots (m_r + 1) d(N) \sqrt{N} + p_1^{m_1} \cdots p_r^{m_r} (m_1 + 1) \cdots (m_r + 1) \\ &\ll p_1^{\frac{3m_1}{2}} \cdots p_r^{\frac{3m_r}{2}} \log (4(p_1^{m_1} \cdots p_r^{m_r})) d(N) \sqrt{N} (m_1 m_2 \cdots m_r), \end{aligned}$$

where $d(N)$ is the divisor function. □

4. THE MEASURE μ

Following the definition from Section 2, for non-negative integers m_1, m_2, \dots, m_r , we get,

$$\begin{aligned} C_{\underline{m}} &:= \lim_{s(N,k) \rightarrow \infty} \frac{1}{(2s(N,k))^r} \sum_{\substack{n \\ 1 \leq i \leq n}} e(\pm m_1 \theta_1^{(n)}(p_1) \pm \cdots \pm m_r \theta_r^{(n)}(p_r)) \\ &= \prod_{i=1}^r \left(\lim_{s(N,k) \rightarrow \infty} \frac{1}{(2s(N,k))^r} \sum_{i_n} e(\pm m_i \theta_i^{(n)}(p_i)) \right) \\ &= \prod_{i=1}^r c_{m_i}, \end{aligned}$$

where c_{m_i} are the m_i -th Weyl limit of the family $\left\{ \frac{\pm \theta_i^{(n)}}{2\pi} \right\}$ and from Theorem 18 of [8], we have

$$c_{m_i} = \begin{cases} 1 & \text{if } m_i = 0, \\ \frac{1}{2} \left(\frac{1}{p_i^{m_i}} - \frac{1}{p_i^{m_i-2}} \right) & \text{if } m_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if m_1, m_2, \dots, m_r are all zero then $C_{0,\dots,0} = 1$. Define the measure

$$\mu = F(-x_1, \dots, -x_r) dx_1 \cdots dx_r,$$

where

$$\begin{aligned} F(x_1, \dots, x_r) &= \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_r=-\infty}^{\infty} c_{m_1} \cdots c_{m_r} e(m_1 x_1) \cdots e(m_r x_r) \\ &= F(x_1) \cdots F(x_r) \\ &= \frac{4(p_1 + 1) \sin^2 2\pi x_1}{(p_1^{\frac{1}{2}} + p_1^{-\frac{1}{2}})^2 - 4 \cos^2 2\pi x_1} \cdots \frac{4(p_r + 1) \sin^2 2\pi x_r}{(p_r^{\frac{1}{2}} + p_r^{-\frac{1}{2}})^2 - 4 \cos^2 2\pi x_r}. \end{aligned}$$

The above $F(x_1, \dots, x_r) dx_1 \cdots dx_r$ determines a measure on $[0, 1]^r$ and is the distribution function for the tuples of numbers

$$(x_{i_1}, \dots, x_{i_r}) = \left(\pm \frac{\theta_1^{(n)}(p_1)}{2\pi}, \dots, \pm \frac{\theta_r^{(n)}(p_r)}{2\pi} \right).$$

The measure giving the distribution of $(\cos \theta_1^{(n)}(p_1), \dots, \cos \theta_r^{(n)}(p_r))$ is therefore

$$\begin{aligned} &= F\left(\frac{\cos^{-1} x_1}{2\pi}, \dots, \frac{\cos^{-1} x_r}{2\pi}\right) d\left(\frac{\cos^{-1} x_1}{2\pi}\right) \cdots d\left(\frac{\cos^{-1} x_r}{2\pi}\right) \\ &= F\left(\frac{\cos^{-1} x_1}{2\pi}\right) d\left(\frac{\cos^{-1} x_1}{2\pi}\right) \cdots F\left(\frac{\cos^{-1} x_r}{2\pi}\right) d\left(\frac{\cos^{-1} x_r}{2\pi}\right) \\ &= \frac{2(p+1)}{\pi} \frac{\sqrt{1-x_1^2}}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - 4x_1^2} dx_1 \cdots \frac{2(p+1)}{\pi} \frac{\sqrt{1-x_r^2}}{(p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 - 4x_r^2} dx_r. \end{aligned}$$

Thus, the distribution of the tuples of numbers

$$(2 \cos \theta_1^{(n)}(p_1) \cdots 2 \cos \theta_r^{(n)}(p_r))$$

is given by $\prod_{i=1}^r \mu_{p_i}$ after an easy change of variable.

5. CHEBYCHEV POLYNOMIALS AND THE TRACE OF HECKE OPERATORS

For any integer $n \geq 0$, the n -th Chebychev polynomial $X_n(x)$ is defined as follows:

$$X_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad \text{where } x = 2 \cos \theta.$$

Serre proved the following result in [11].

Lemma 8. *We have $T'_{p^m} = X_m(T_p)$.*

From [8, page 697], when $m_i = 1$, we have,

$$\sum_{n=1}^{s(N,k)} 2 \cos \theta_i^{(n)}(p_i) = \text{Tr}(T'_{p_i}).$$

Since, for all integers $m \geq 2$,

$$\sum_{n=1}^{s(N,k)} 2 \cos m\theta = X_m(2 \cos \theta) - X_{m-2}(2 \cos \theta),$$

we have for $m_i \geq 2$,

$$(2) \quad \sum_{n=1}^{s(N,k)} 2 \cos \theta_i^{(n)}(p_i) = \text{Tr}(T'_{p_i^{m_i}}) - \text{Tr}(T'_{p_i^{m_i-2}}).$$

Now we prove the following theorem,

Theorem 9. *For all non zero positive integers m_1, m_2, \dots, m_r consider $\underline{m} = (m_1, m_2, \dots, m_r)$, the Weyl limits $C_{\underline{m}}$ are given by*

$$C_{\underline{m}} = \begin{cases} 1 & \text{if } m_1 = \dots = m_r = 0, \\ \prod_{i=1}^r \left(\frac{1}{p_i^{\frac{m_i}{2}}} - \frac{1}{p_i^{\frac{m_i-2}{2}}} \right) & \text{if } m_i, i = 1, 2, \dots, r \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\left| \prod_{i=1}^r \sum_{n=1}^{s(N,k)} 2 \cos m_i \theta_i^{(n)}(p_i) - s(N,k) C_{\underline{m}} \right| \ll p_1^{\frac{3m_1}{2}} \dots p_r^{\frac{3m_r}{2}} (m_1 \dots m_r) \log(4(p_1^{m_1} \dots p_r^{m_r})).$$

Proof. For any integers $m_1, m_2, \dots, m_r \geq 2$, using (2), we have

$$\begin{aligned} (3) \quad & \prod_{i=1}^r \sum_{n=1}^{s(N,k)} \{2 \cos m_i \theta_i^{(n)}(p_i)\} \\ &= \prod_{i=1}^r \left(X_{m_i}(2 \cos \theta_i^{(n)}(p_i)) - X_{m_i-2}(2 \cos \theta_i^{(n)}(p_i)) \right) \\ &= \sum_{b_1, b_2, \dots, b_r \in Z} \prod_{i=1}^r X_{b_i}(p_i) (2 \cos \theta_i^{(n)}(p_i)), \end{aligned}$$

where $Z = \{m_1, m_2, \dots, m_r, m_1 - 2, m_2 - 2, \dots, m_r - 2\}$. We know that if a linear operator T is diagonalizable and λ is an eigenvalue of T , then, for any polynomial $P(x)$, the eigenvalue of $P(T)$ is $P(\lambda)$. Since the Hecke operators T_m and T_n commutes with each other, there exists an ordered basis such that every Hecke operator can be represented by a diagonal matrix with

respect to the basis. Using all the above facts, (3) equals

$$\begin{aligned}
& \sum_{b_1, b_2, \dots, b_r \in Z} \text{Tr}(T'_{p_1^{b_1} \dots p_r^{b_r}}) \\
& \ll \sum_{b_1, b_2, \dots, b_r \in Z} (p_1^{b_1} \dots p_r^{b_r}) \left(\frac{k-1}{12} \right)^r \\
& \quad + p_1^{\frac{3m_1}{2}} \dots p_r^{\frac{3m_r}{2}} \log 4(p_1^{m_1} \dots p_r^{m_r})(m_1 m_2 \dots m_r) \\
& \ll \left(\frac{k-1}{12} \right)^r \prod_{n=1}^r \left(\frac{1}{p_n^{\frac{m_n}{2}}} - \frac{1}{p_n^{\frac{m_n-2}{2}}} \right) \\
& \quad + p_1^{\frac{3m_1}{2}} \dots p_r^{\frac{3m_r}{2}} \log 4(p_1^{m_1} \dots p_r^{m_r})(m_1 m_2 \dots m_r) \\
(4) \quad & \ll \left(\frac{k-1}{12} \right)^r \prod_{n=1}^r \left(\frac{1}{p_n^{\frac{m_n}{2}}} - \frac{1}{p_n^{\frac{m_n-2}{2}}} \right) |\text{Tr}(T'_{p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}})|.
\end{aligned}$$

Now

$$\begin{aligned}
& \left| \prod_{i=1}^r \sum_{n=1}^{s(N,k)} 2 \cos m_i \theta_i^{(n)}(p_i) - s(N, k) C_{\underline{m}} \right| \\
& = \left| \sum_{j=1}^4 B_j(m_1 \dots m_r) - s(N, k) C_{\underline{m}} \right| \\
& \leq |B_1 - s(N, k) C_{\underline{m}}| + \left| \sum_{j=2}^4 B_j(m_1 \dots m_r) \right|.
\end{aligned}$$

Using (4) and the fact that $s(N, k) C_{\underline{m}}$ behaves like $B_1(m_1 \dots m_r)$, we have

$$\begin{aligned}
& \left| \prod_{i=1}^r \sum_{n=1}^{s(N,k)} 2 \cos m_i \theta_i^{(n)}(p_i) - s(N, k) C_{\underline{m}} \right| \\
& \ll p_1^{\frac{3m_1}{2}} \dots p_r^{\frac{3m_r}{2}} (m_1 \dots m_r) \log (4(p_1^{m_1} \dots p_r^{m_r})).
\end{aligned}$$

□

6. EFFECTIVE VERSIONS

To prove the effective result, we use the following variant of the Erdős-Turán inequality that can be found in [6, Proposition 7.1].

Theorem 10. *For any $S = \prod_{n=1}^r [\alpha_n, \beta_n] \subset [0, \frac{1}{2}]^r$ and a sequence of tuples of number $\{(x_{m_1}, \dots, x_{m_r})\} \in [0, \frac{1}{2}]^r$, we define*

$$N_S(V) := \#\{n \leq V : (x_{m_1}, \dots, x_{m_r}) \in S\}$$

and

$$D_S(V) := |N_S(V) - V\mu(S)|.$$

For $I = \prod_{n=1}^r [\alpha_n, \beta_n]$, we have

$$D_I(V) \leq \sum_{m_1, m_2, \dots, m_r \in ([-m, m] \cap \mathbb{Z})^r} w(m_1, \dots, m_r) \Delta((m_1, \dots, m_r), V) \\ + 10 \frac{2}{M+1} \sum_{1 \leq m \leq M} \max_{1 \leq t \leq r} \Delta_t(m, V) + 12 \|F\| \frac{2V}{M+1}$$

for any integers $V, M \geq 1$, where

$$\Delta_t(m, V) = \left| \sum_{n \leq V} \cos(2\pi m x_i) - V c_m \right|, \quad (\text{for } m \in \mathbb{Z}),$$

$$|\Delta((m_1, \dots, m_r), V)| = \left| \sum_{m_1, \dots, m_r \leq V} \cos(2\pi m_1 x_1) \cdots \cos(2\pi m_r x_r) \right|,$$

for $(m_1, \dots, m_r) \in \mathbb{Z}^r$ and

$$w(m_1, \dots, m_r) = \left((2\pi)^r \prod_{t=1}^r \min \left(\frac{1}{\pi |m_t|}, \beta_t - \alpha_t \right) + \frac{2}{M+1} \right),$$

and

$$\|F\| = \max_{1 \leq t \leq r} \|F_t\|_\infty + \prod_{t=1}^r \|F_t\|_\infty$$

with $\|F_t\|_\infty = \max_{x \in [0,1]} |F_t(x)|$.

Proof of Theorem 5.

Choose $\theta_i^{(n)}(p_i) \in [0, \frac{1}{2}]$ such that $2 \cos 2\pi \theta_i^{(n)}(p_i) = a_i^{(n)}(p_i)$. Given any sub interval $I \subset [-2, 2]$ choose sub interval $I' \subset [0, \frac{1}{2}]$ so that $\theta_i^{(n)}(p_i) \in I'$ if and only if $a_i^{(n)} \in I$. By Theorem 10, we have

$$\# \{1 \leq n \leq (s(N, k)) : (a_1^{(n)}(p_1), \dots, a_r^{(n)}(p_r)) \in I - V\mu(I) \mid \\ \ll \frac{s(N, k)r}{M+1} + \left| \prod_{i=1}^r \sum_{n=1}^{s(N, k)} 2 \cos m_i \theta_i^{(n)}(p_i) - s(N, k) C_m \right|.$$

Using Proposition 7, we get

$$\ll \frac{s(N, k)r}{M+1} + (p_1^{\frac{3m_1}{2}} \cdots p_r^{\frac{3m_r}{2}}) (m_1 \cdots m_r) \log(4(p_1^{m_1} \cdots p_r^{m_r})) \log M^r.$$

Since the main contribution is coming from $\left(p_1^{\frac{3m_1}{2}} \cdots p_r^{\frac{3m_r}{2}}\right)$, by choosing

$$M = \frac{\log kN}{\log p_1 p_2 \cdots p_r}, \text{ we get the required result.} \quad \square$$

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